

Fall of an Elastic Bar in Central Gravitational Field: I. Newtonian Gravity

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Abstract

Within some reasonable approximations we calculate deformation of an elastic bar, falling on a source of central gravitational field. We consider both elastic deformations and plastic flow together with destruction of the bar. Concrete calculations for a number of materials are presented.

KEY WORDS: Tidal forces, elastic bar, deformation, plastic flow, elasticity limit, destruction.

1 Introduction

The study of the extended deformable bodies motion in general relativity (GR) is an important and technically challenging problem with fundamental as well as practical applications. In order to be consistent with principles of GR the solution requires a thorough review of all the notions which are commonly used in nonrelativistic continuous media physics. General principles lying at the foundation of deformable bodies physics were actively debated when the relativity theory first appeared [1, 2, 3]. Contemporary understanding of the problem is based upon seminal works [4, 5, 6]. Different aspects of the elasticity theory in GR were further developed in works [7] (gauge formulation), [8] and [9, 10, 11] (general theorems of existence and uniqueness). The most important and widely accepted topics in the field are: *material (frame independent) description of elastic medium*, *1+3-decomposition of space-time* and *Lagrangian formulation of gravitoelastic equations*.

Note that more detailed treatment and application of these topics gave rise to slightly different versions of the general approach, depending on author's conceptions; some of these conceptions are outside the scope of elasticity in GR [12, 13, 18]. Despite that these slightly different approaches to the problem

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reproduce the right nonrelativistic limit, the predictions regarding experiments with extended elastic bodies based on them can diverge drastically. That applies to the behavior of the bodies in strong gravitational fields (horizon of black hole) and even in weak fields without Newtonian limit (gravitational waves). So, the study of the behavior of deformable bodies in such exotic situations (from the point of view of Newtonian gravity) could serve for improvement and further development of elasticity theory in GR. At present both gravitational waves physics and black hole physics are theoretically well founded and actively developing topics. However, while the former has many intersections with elasticity in GR (first of all bar detectors and their interaction with wave perturbation in the space-time metric should be mentioned), incorporation of elasticity ideas into the black holes physics is much more modest. The majority of attempts in this or related fields dealt with the self-consistent problem of equilibrium of astronomical bodies that possess elastic properties (among the last works on the subject is already cited [13]) and with tidal phenomena in celestial mechanics and astrophysics [14, 15, 16, 17].

The present paper can be treated as a small step towards understanding what will be the influence of a strong gravitational field on test deformable bodies. Note that this problem is not the subject of traditional elasticity GR since strong gravitational field causes tidal stresses which could lead to plastic flow and mechanical destruction of any real body. We are going to illustrate our approach by a relatively simple example of a one-dimensional continuous medium that is thin elastic bar, falling towards the source of central gravitational field. We'll show that our treatment of the problem encounters two difficulties which can be analyzed independently. The first difficulty originates from nonlinearity of the equations of motion and has nothing to do with relativistic aspects of the problem whatsoever. The second difficulty arises from dealing with relativistic properties of both motion and space-time that have to taken into account in order to get a correct quantitative picture of deformations.

The aim of this part of the paper is to solve the first kind of difficulties within Newtonian gravity and to lay a foundation for more precise and consistent consideration in the context of GR. We formulate an approximate "quasistatic fall" approach which allows to solve nonlinear equations by the perturbations method, with smallness parameter L/r , where L is the undeformed bar's length, r is instant distance from the bar to the source of gravitational field. The law of motion of point-like particles will play the role of "null approximation". This nonrelativistic approach allows us to calculate the deformational characteristics of bars up their destruction. In section 2 we derive the exact equation of motion (within Newtonian theory) that describes the bar's behaviour including the deformation. In section 3 we discuss the exact formulation of the quasistatic fall condition. We find out that it is valid for bars made of solid materials. In section 4 we derive an approximate equation of motion in first order on L/r and find solution to the derived equation that satisfies all necessary initial and boundary conditions. Our approximate solution for elasticity reproduces the exact static solution (see [19]) after the transversal averaging and coarsening procedures. Quantitative and qualitative deformational characteristics of the

bar are calculated in section 5. In section 6 we derive slightly simpler formulae for calculation of definite deformational characteristics: the time of the beginning of the plastic flow and the time of destruction as well as the corresponding positions of the bar. Then, using the derived formulae, we find some "universal" relations independent of the bar's initial state and present results acquired for some specified materials. In the next part of the paper there will be presented relativistic analysis of the problem considering a deformable body falling into Schwarzschild black hole.

2 Statement of the problem

Let us consider radial fall of a thin probe bar from an initial position distanced from a massive source of Newtonian central gravitational field. While in motion the bar stays oriented along the line of force (in a radial direction). Let us put the origin O of an one-dimensional coordinate system at the force center and direct the coordinate axe OX along the bar's motion radial line. Then kinematics of the bar can be described by a displacement field $x(t, \xi)$, defined on the points ξ of a remote unstrained bar (Lagrangian picture)¹, and depending on time. The initial and boundary conditions are as follows:

$$x(0, \xi) = r_0 + \xi, \quad \xi \in [-L/2; L/2]; \quad \dot{x}(0, \xi) = -v_0 = \text{const}; \quad \sigma|_{\xi=0, L} = 0. \quad (1)$$

Here r_0 is the initial coordinate for the bar's center of mass, L is the unstrained bar's length, v_0 is absolute value of initial velocity, $\sigma = \sigma(t, \xi) = \sigma_{xx}$ is 1-dimensional "stress tensor" (later to be referred as "stress") that depends on ξ and t . Here and below (if not specified otherwise) the dot denotes differentiation with respect to time coordinate t (or τ — see below), the accent denotes differentiation with respect to space Lagrangian coordinate ξ . To be more precise, we define σ as the following average value:

$$\sigma = \frac{1}{S} \int \sigma_{xx}^{\text{local}}$$

where integral is being taking over transversal section of the bar, S means its area and $\sigma_{xx}^{\text{local}}$ is true longitudinal stress tensor component, that depends on the points of the transversal section.

Lets us consider small element dx of the bar at some fixed moment of time t . Its ends have material coordinates ξ and $\xi + d\xi$. Inertial force, related to the

¹So, we absolutely ignore transversal elasticity of the bar. Such simplifying is correct, since ratio of absolute transversal Δr and longitudinal Δl deformations satisfies the following nonequality:

$$\frac{\Delta r}{\Delta l} \sim \mu \frac{R}{L} \ll 1$$

due to thinness of the bar (R is its characteristic transversal size) and smallness of Poisson's coefficient for majority of solid materials. Exact results of [19] support this assumption: even for solid bar with $R/L = 1/4$ ratio $\Delta r_{\text{max}}/\Delta l \sim 1/10$ near the Earth.

element is expressed by the formula:

$$-dm \ddot{x} = -\rho S \ddot{x} dx,$$

where $\rho = \rho(t, \xi)$ is the local volume mass density of the bar. Gravitational force, acting on the same element, has the form:

$$-\frac{GM dm}{x^2} = -\frac{GM \rho S dx}{x^2},$$

where M is mass of origin, G is the Newtonian constant. Finally, elastic force, acting on the element looks like this:

$$\frac{\partial(\sigma S)}{\partial x} dx.$$

After we've added the three forces together and equated the resultant to zero, the following equation of motion can be obtained:

$$\ddot{x} - \frac{1}{\rho S} \frac{\partial(\sigma S)}{\partial x} = -\frac{GM}{x^2}. \quad (2)$$

In order to exclude stress it is necessary to set some defining relation [20] for the material of the bar. Said relation normally connects stress σ with relative deformation $\epsilon \equiv x' - 1$. In this paper we'll use piecewise linear defining relation, which is commonly applied in the majority of the model problems dealing with elasticity theory and strength of materials [21]. Fig.1 shows a simplified effective diagram "strain-stress" illustrating this defining relation. In real laboratory

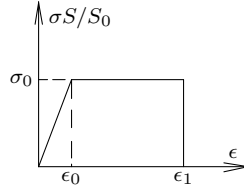


Figure 1: Simplified stress diagram of bar's material, expressing results of laboratory experiments with real bar.

experiments with bars the controlled external force is usually related to the initial area S_0 of the transversal section before stress starts to build up. So, such experiments allow only to determine dependency of $\sigma S/S_0$ on ϵ , which includes variation of the transversal section area under deformation. The inclined part of the diagram describes elastic deformations (*Hooke's law*), for which $\sigma S/S_0 = E\epsilon$, where $E = \sigma_0/\epsilon_0$ is Young's modulus for the material of the bar, ϵ_0 is limit of elasticity, σ_0 is stress of plastic flow. The latter is described by the horizontal part of the diagram, where strain increases under practically constant stress. Parameter ϵ_1 characterizes ultimate relative deformation causing the destruction of the material.

Using the chosen defining relation, the term linked to stress in (2) can be transformed (under elastic deformations) as follows:

$$-S_0 \frac{\partial(E\epsilon)}{\partial x} = -S_0 \frac{(E(x' - 1))'}{\partial x / \partial \xi},$$

where in the differentiation with respect to x we've turned to the Lagrange variable ξ . Assuming that the bar is homogeneous, we have $E = \text{const}$ and $\rho_0 = \rho(0, \xi) = \text{const}$. Using of the relation:

$$\rho_0 S_0 = \rho(t, \xi) x' S,$$

that expresses the law of mass conservation as applied to any element of the bar related to fixed material particles, equation (2) can be rewritten in the form:

$$\square x = -\frac{r_s}{2x^2}, \quad (3)$$

where $\square \equiv \partial_\tau^2 - \partial_\xi^2$ is the two-dimensional D'Alembert operator, $\tau = c_0 t$, $c_0 = \sqrt{E/\rho_0}$ is sound velocity in the material of the bar, $r_s = 2GM/c_0^2$ is sound analog of gravitational radius, which we'll call the *sound radius*².

Equation (3) is nonlinear wave equation, for which standard methods of mathematical physics are inapplicable. It easy to find a family of particular soliton-like solutions, which is characterized by special dependency on ξ and τ : $x = x(\zeta)$, $\zeta = a\tau + b\xi$, where a and b are arbitrary constants, satisfying the condition: $|a| \neq |b|$. Substitution to (3) leads to an ordinary differential equation:

$$x'' + \frac{k}{x^2} = 0, \quad (4)$$

with $k = r_s/2(a^2 - b^2)$, where the accent denotes differentiation with respect to ζ . However, its general solution:

$$-\frac{1}{A} \sqrt{(Ax + 2k)x} + \frac{k}{A^{3/2}} \ln(k + Ax + \sqrt{Ax(Ax + 2k)}) = \zeta - \zeta_0, \quad (5)$$

where A and ζ_0 are integration constants, doesn't satisfy conditions (1), no matter what constants or parameters are being chosen. In general, our study will not involve those stressed states of the bar that are associated with running waves. We'll restrict ourselves to seeking solutions to the equation (3) that describe a smooth (quasi-static) variation of the bar's stressed state from null strain (far from the gravitational center) up to the point of destruction. Note, that for those parts of the bar, where elasticity limit is exceeded, the equation (3), according to the stress diagram in Fig.1, will acquire a simpler form as that of an ordinary differential equation similar to (4).

²In a difference with gravitational one, it depends on elastic properties of bodies, falling on the gravitational center.

3 Condition of quasi-static fall

On an obvious assumption that the law of motion for material points roughly applies to the motion of extended small-sized bodies, an approximate solution to the equation (3) can be constructed. To be more precise, we state that:

$$\lim_{L \rightarrow 0} \max_{t \in [0; T]} |x(t) - x_0(t)| = 0, \quad (6)$$

where L is characteristic size of the extended body, $x(t)$ is exact law of motion for the body's center of mass, calculated for time interval $[0; T]$ with regard to its extended structure, $x_0(t)$ is law of motion for point-like body with the same integral characteristic (mass, charge etc.), calculated for the same time interval. The relation (6) is, in fact, one of the basic statements of both classical mechanics and relativity theory in implicit form. (Precise formulations and important rigorous theorems see in [10]).

This assumption allows us to seek solution to the equation (3) as follows:

$$x(\tau, \xi) = x_0(\tau) + \xi + \chi(\tau, \xi), \quad (7)$$

where $x_0(\tau) + \xi$ describes the motions of the rigid bar (tidal forces to be disregarded), which center of mass initially is in the position $x_0(0) = r_0$, while correction $\chi(\tau, \xi)$ describes strain of the bar, subjected to in some set mode of motion $x_0(\tau)$. After we've calculated the strain in a set mode of motion, if necessary, we can define the mode of fall more accurately and recalculate the strain etc.

In order to establish the limits of validity of our approach, let us estimate critical distance r_c , over which the bar is destroyed. Upon equating the characteristic tidal stress to ultimate stress, which is for majority of solid materials three orders less then their Young's modulus, our estimation appears as follows:

$$r_c \sim 10(r_g L^2)^{1/3} (c/c_0)^{2/3},$$

where $r_g = 2GM/c^2$ is the gravitational radius of the field's source. Assuming that $c/c_0 \sim 10^5$, we obtain a more convenient formula for calculating the estimations:

$$r_c \sim 10^4 (r_g L^2)^{1/3}. \quad (8)$$

It follows from (8) that for gravitational centers whose mass is $M \leq 10^3 M_\odot$ and bars of the lengths $L \sim 10^0 \text{m}$ $r_c > r_g$ (for the Sun $r_g \sim 1 \text{km}$ and $r_c \sim 100 \text{km}$).

The validity condition of the approach discussed above is the "quasi-static fall" quality: time ΔT for characteristic variation of the tidal force must exceed by far the time L/c_0 for the longitudinal sound waves propagation along the bar. The time ΔT can be roughly estimated as $x_0/c_0 \dot{x}_0$. As the expression implies, the time will be minimal at the moment of destruction. The energy conservation law has the following dimensionless form:

$$\dot{x}_0^2 = \frac{r_s}{x_0} + \frac{v_0^2}{c_0^2} - \frac{r_s}{r_0} \quad (9)$$

The first term on the right at the moment of destruction has the order 10^7 (when $L \sim 10^0\text{m}$, $M \sim M_\odot$), while the second and third under nonrelativistic initial velocities and large enough (compared to the gravitational radius of the source) initial distances have considerably less order. Thus we can disregard them at the moment of destruction and applying (8) we get:

$$\Delta T_{\min} \sim 10^{3/2} \frac{L}{c_0} > \frac{L}{c_0}. \quad (10)$$

So, for bars made of solid materials (metals, steels and alloys, for which $\sigma_1 \sim 10^{-3}E$ and $c_0 \sim 10^{-5}c$) the condition of the fall's quasi-static quality is met automatically by application of its law of motion. For softer materials the condition can be violated near the point of destruction. That implies the necessity of studying the shock waves and sound retarding impact on the problem.

4 Approximate solution

After we have decomposed the gravitational force in the row (with respect to L/x_0) near the rigid bar's center of mass (its instant position) $x_0(\tau)$, the right-hand side of the equation (3) can be rewritten in the following form:

$$-\frac{r_s}{2(x_0 + \xi)^2} = -\frac{r_s}{2x_0^2} + \frac{r_s \xi}{x_0^3} + o\left(\frac{L}{x_0}\right). \quad (11)$$

After substituting it into (3) and taking (7) into account, we obtain the following approximate equation of motion:

$$\ddot{x}_0 + \ddot{\chi} - \chi'' = -\frac{r_s}{2x_0^2} + \frac{r_s \xi}{x_0^3}.$$

The first terms on the left and on the right are cancelled as the definition of $x_0(\tau)$ implicates. The second term on the left is responsible for *back reaction shown by motion on the deformation of the bar*. The term is of the order $o(L^2/x_0^2)$ that can be proven later when the approximate solution for χ is obtained, meaning that we can disregard it for the time being. Then the equation that define strains, takes the form:

$$\chi'' = -\frac{r_s \xi}{x_0^3}.$$

Its general solution is:

$$\chi = -\frac{r_s \xi^3}{6x_0^3} + C_1(\tau)\xi + C_2(\tau),$$

where C_1, C_2 are arbitrary functional constants of integration. Solution $x(\tau, \xi)$, that satisfies both initial (with accuracy L^3/r_0^3) and boundary conditions (1), takes the form:

$$x(\tau, \xi) = x_0(\tau) + \xi \left(1 + \frac{r_s}{2x_0^3} \left(\frac{L^2}{4} - \frac{\xi^2}{3}\right)\right). \quad (12)$$

5 Elastic and plastic deformations pictures of the falling bar

Upon defining the displacements field $u(\tau, \xi)$ by the formula: $x = x_0 + \xi + u(\xi)$, the solution (12) results in expression:

$$u(\tau, \xi) = \frac{r_s \xi}{2x_0^3} \left(\frac{L^2}{4} - \frac{\xi^2}{3} \right). \quad (13)$$

Fig.2 shows the displacement diagram at some arbitrary (but preceding the plastic flow) moment of time.

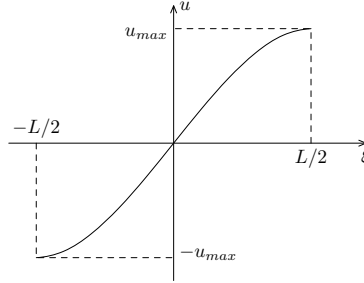


Figure 2: Instant displacements diagram related to the center of the bar. $u_{\max} = r_s L^3 / 24x_0^3$.

The diagram during the fall is stretched in vertical direction. The instant effective stress diagram, expressed by the formula:

$$\sigma_{\text{eff}} \equiv \frac{\sigma S}{S_0} = \frac{r_s E}{2x_0^3} \left(\frac{L^2}{4} - \xi^2 \right) \quad (14)$$

is shown in Fig.3. As the previous diagram, this one shows tendency to vertical

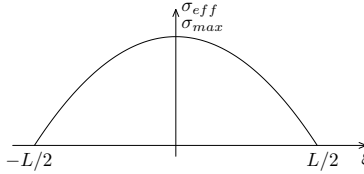


Figure 3: Instant effective stress diagram before plastic flow. $\sigma_{\max} = Er_s L^2 / 8x_0^3$.

stretching of the same time law. At the moment τ^* , defined by relation:

$$x_0(\tau^*) = \left(\frac{r_s L^2}{8\epsilon_0} \right)^{1/3} \quad (15)$$

stress reaches the first stress limit σ_0 at the middle point of the bar and this marks the nucleus of the plastic phase. The size $2\xi_b$ of the part of the bar engaged in the plastic phase increases in time symmetrically in both directions from the center of the bar according to the expression:

$$\xi_b(\tau) = \sqrt{\frac{L^2}{4} - \frac{2\sigma_0 x_0^3(\tau)}{r_s E}}. \quad (16)$$

This dependency is shown in Fig. 4.

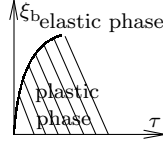


Figure 4: Boundary of elastic and plastic phases in falling bar as function of time.

When the front of the plastic phase part approaches the ends of the bar, its velocity $d\xi_b/dt$ asymptotically tends to zero. Any element of the bar in the plastic phase (i.e. situated inside the interval $[-\xi_b(\tau), \xi_b(\tau)]$ at the moment τ) moves free in gravitational field, as the stress diagram gives $\sigma' = 0$ for any point of plastic phase.

Since the appearance of the plastic phase, matter elements situated in the vicinity of the middle point of the bar, continue to suffer the stretch strain. However, they doesn't stretch the same way the parts of elastic phase do (i.e. in accordance to the solution (12)), but as particles of noninteracting "dust packet". It is in the central point $\xi = 0$ the relative plastic deformation will reach its maximum ϵ_1 at the moment τ^{**} , causing the bar to break in two halves.

In order to calculate the relative deformation at the middle point of the bar at an arbitrary moment of the plastic phase, it is worth noting that the element $d\xi$ of the bar containing middle point, will by the beginning of the plastic phase acquire the length $ds = (\epsilon_0 + 1) d\xi$. Its further stretching can be described by the equation:

$$ds' = \left. \frac{\partial \bar{x}_0(\tau, s)}{\partial s} \right|_{s=0} ds = \left. \frac{\partial \bar{x}_0(\tau, s)}{\partial s} \right|_{s=0} (\epsilon_0 + 1) d\xi.$$

Here $\bar{x}_0(\tau, s)$ means a family of solutions, that describe free particles with coordinates $x_0(\tau^*) + s$ at the moment $\tau = \tau^*$ and velocity $\dot{x}_0(\tau^*)$. Relative deformation with respect to initial configuration is given by the expression:

$$\epsilon' = \frac{ds'}{d\xi} - 1 = \left. \frac{\partial \bar{x}_0(\tau, s)}{\partial s} \right|_{s=0} (\epsilon_0 + 1) - 1.$$

Then equation for τ^{**} takes the form:

$$\left. \frac{\partial \bar{x}_0(\tau^{**}, s)}{\partial s} \right|_{s=0} = \frac{\epsilon_1 + 1}{\epsilon_0 + 1}. \quad (17)$$

Equations (13)-(17) together with the solution (5) for free point-like particles give a complete picture of deformations of the bar up to its destruction.

6 Design formulae and calculations

Before we start deriving design formulae for calculations we want to stress that any more or less distinctive relative deformations of bar (say, about 0.01%) appears only at the latest times, when the bar moves considerably close to the gravitational center. So, for the source of mass about of the Sun's and the bar about 1m long we can expect by (14) the deformation $\bar{\epsilon} = 10^{-4}$ only at distance of about $\bar{r} \sim 100\text{km}$ from the gravitational center. The general formula is:

$$\bar{r} \sim \left(\frac{r_s L^2}{\bar{\epsilon}} \right)^{1/3}$$

It gives us practically the same result in the majority of the other reasonable situations. So, the largest and most important part of the bar's deformational history takes place under $x \lesssim \bar{r}$. It implies, that instead of the general solution (5) we can use its simpler asymptotic form under small x , where the law of motion is parabolic:

$$x_0(\tau) = \left(\frac{9r_s}{4} \right)^{1/3} (\tau_0 - \tau)^{2/3} = \left(\frac{9r_g c^2}{4} \right)^{1/3} (t_0 - t)^{2/3}, \quad (18)$$

where $t_0 = \tau_0/c_0$ means total time of particle's fall. This is an unique value, which depends on the initial state of the falling object. It can be calculated by means of the exact formula (5):

$$t_0 = \frac{\sqrt{GM r_0/2}}{\varepsilon_0} \phi(\varepsilon_0 r_0/GM), \quad (19)$$

where $\varepsilon_0 = E_0/m$ is initial specific energy of the falling object, r_0 is initial distance from the gravitational center and the function

$$\phi(z) = \sqrt{z+1} - \frac{1}{2\sqrt{z}} \ln(1+2z+2\sqrt{z(1+z)}).$$

For $r_0 \sim 1$ a.u., $M \sim M_\odot$ and $v = 50\text{km/s}$ the formula (19) gives us $t_0 \sim 25$ days. The exact validity criterion for the parabolic law of motion near the source is as follows:

$$\frac{\varepsilon_0}{c_0^2} \ll \left(\frac{r_s}{L} \right)^{2/3} \bar{\epsilon}^{1/3}.$$

It is satisfied when the initial velocities are nonrelativistic and initial distances exceed \bar{r} by far.

By using (4) and general formulae (13)-(17) of the previous section, it is easy to derive the following convenient design expressions, that describe all characteristic events of the bar's deformational history:

$$\tau^* = \tau_0 - \frac{L}{3\sqrt{2\epsilon_0}}; \quad \xi_b(\tau) = \sqrt{\frac{L^2}{4} - \frac{9\epsilon_0}{2}(\tau_0 - \tau)^2}. \quad (20)$$

Derivation of (17) is slightly more complicated. Decomposition of (5) in the vicinity of the parabolic law has the following form:

$$C - \tau = \int \frac{dx}{\sqrt{A + r_s/x}} = \frac{2x^{3/2}}{3\sqrt{r_s}} - \frac{Ax^{5/2}}{5r_s^{3/2}} + o(A). \quad (21)$$

Here C and A are integration constants determined by initial conditions at the moment of the appearance of the plastic phase:

$$x(\tau^*) = x_0(\tau^*) + s; \quad \dot{x}(\tau^*) = \dot{x}_0(\tau^*). \quad (22)$$

Upon substituting (21) in (22) and solving equations with respect to C and A (in the first order with respect to s) we obtain the following implicit equation that defines the law of motion in s -vicinity of the center $\xi = 0$ after the moment of the appearance of the plastic phase:

$$\frac{2x^{3/2}}{3\sqrt{r_s}} - \frac{sx^{5/2}}{5\sqrt{r_s}x_0^{*2}} - \frac{4}{5}s\sqrt{\frac{x_0^*}{r_s}} = \tau_0 - \tau. \quad (23)$$

Here $x_0^* \equiv x_0(\tau^*)$. After differentiating of the expression with respect to s under fixed τ and assuming $s = 0$, we obtain:

$$\left. \frac{dx}{ds} \right|_{s=0} = \frac{4}{5}\sqrt{\frac{x_0^*}{x}} + \frac{1}{5}\frac{x^2}{x_0^{*2}}.$$

Substituting in this expression $x(\tau) = x_0(\tau)$, we finally get:

$$\left. \frac{dx}{ds} \right|_{s=0} = \frac{4}{5}\sqrt{\frac{x_0^*}{x_0}} + \frac{1}{5}\frac{x_0^2}{x_0^{*2}}. \quad (24)$$

Destruction conditions (17) and (24) allow us to come up with the following formula for the time of destruction:

$$\tau^{**} = \tau_0 - \frac{L\eta^3}{3\sqrt{2\epsilon_0}}, \quad (25)$$

where η is root of equation:

$$\frac{4}{\eta} + \eta^4 = \frac{5(\epsilon_1 + 1)}{\epsilon_0 + 1}.$$

The dependency of all obtained formulae (20)-(25) on initial conditions is based on the parameter τ_0 only. Some of consequences of these formulae are

perfectly independent of the initial conditions. For example, full time of plastic phase existence:

$$\Delta\tau \equiv \tau^{**} - \tau^* = \frac{L}{3\sqrt{2\epsilon_0}}(1 - \eta^3) \quad (26)$$

and the size of the plastic phase at the moment of destruction:

$$2\xi_b(\tau^{**}) = 2\xi_{b\max} = L\sqrt{1 - \eta^6}$$

depend only on the elastic constants for the material of the bar. Also we have the following universal ratio of coordinates $x^{**} \equiv x(\tau^{**})$ and $x^* \equiv x(\tau^*)$:

$$x^{**} = \eta^2 x^*. \quad (27)$$

Table 1 shows results of calculations made for the bars of fixed length and of different solid materials.

Material	$\epsilon_0, 10^{-2}\%$	$\epsilon_1, \%$	x^*/r_g	η	x^{**}/r_g	$2\xi_{b\max}/L$	$\Delta\tau/(\tau_0 - \tau^*)$
Aluminium	3.1	45	53	0.56	17	0.984	0.82
Iron	8.5	50	40	0.54	12	0.987	0.84
Gold	5.0	40	86	0.58	29	0.981	0.80
Copper	5.8	60	54	0.50	13.5	0.992	0.88
Platinum	4.4	45	74	0.56	23	0.984	0.82
Lead	3.1	50	142	0.54	41	0.987	0.84
Silver	3.2	45	82	0.56	26	0.984	0.82
Titanium	9.1	70	39	0.47	9	0.996	0.90
Steel 15GS (small strength)	17.5	18	31	0.78	19	0.880	0.53
Steel N18K9M5T (of high strength)	97	8	17	0.90	14	0.684	0.27
Steel 30H13 (martensite)	75	6	19	0.92	16	0.627	0.22
Alumalloy (Al-Cu-Mg)D19T	49	10.5	31	0.86	23	0.772	0.36
Titanalloy (Ti-Al-V-Cr)TS6	108	4	20	0.95	18	0.514	0.14

Table 1. Results of approximate calculations by formulae (20) and (25) for central body with $M = 2 \cdot 10^{30}$ kg and bars of length $L = 1$ m made of different solid materials. Limit of elasticity ϵ_0 calculated by the formula: $\epsilon_0 = \sigma_{0.2}/E$, where $\sigma_{0.2}$ is (effective) stress which causes residual relative deformation 0.2%. Ultimate deformation $\epsilon_1 = \epsilon_0 + \delta$, where δ is relative residual stretching after the destruction of the bar. Steels labels corresponds to Russian standards [22].

From (15) and (27) it follows that immediately after destruction the plastic phase disappears in both halves under the condition $\eta > 1/2^{1/3} \approx 0.794$. Fifth column of the Table 1 shows that for all listed materials except high alloy steels, the plastic phase will reappear in future generations of the bar's fragments.

Inside of the bars made of high alloy steel, plastic phase appears, increases, disappears and after the fission reappears again in fragments some time later. According to the first formula of (20), bars with lengths of about 10^0cm or less begins to flow near the gravitational radius of the central body, which one should consider already a black hole. More precise and consequent calculations in the context of GR will be presented in next part of the paper.

7 Conclusion

We have considered intermediate nonrelativistic problem concerning the fall of an elastic bar on massive source of central gravitational field. Within the quasi-static fall approach formulae (20) and (25) we've been able to acquire reasonable and in some sense exhaustive description of the bar's deformational history up to its destruction. Note that our simple estimations have shown that for bars made of solid materials like solid metals, steels and alloys the quasistatic fall condition is satisfied automatically. Assumptions used for derivation of (10) from (9) concern general nonrelativistic approximations of the problem.

Results of calculations, presented in Table 1 show that bars with length about 1m falling on source with mass about M_\odot are being destroyed far from the gravitational radius (closest (of the materials listed in Table 1) to it comes the titanium bar — about $10r_g$). Note that the time $\Delta\tau$ of the most important part of the bar's deformational history is about 1ms in absolute units. After the first destruction each fragment of the bar gets divided again etc. However, quantitative treatment of the behaviour of the generations of the bar's fragments becomes more complicated, because of non-elastic hysteresis and strain hardening phenomena, that take place in deformed fragments. Taking average values $\bar{\eta} \approx 0.67$ and $\bar{x}^* \approx 53r_g$ from the Table 1, and using the universal formula (27) we obtain:

$$N \sim \log_{\bar{\eta}} \frac{1}{\sqrt{\bar{x}^*/r_g}} \approx 5$$

— a rough estimation of number of the consequent fragments continuing dividing until they reach the gravitational radius. Near the gravitational radius there will be required a cardinal reconsideration of our formulae within the context of GR.

It is interesting, that the bar, falling in central gravitational field is, in some sense, a perfect experimental device for obtaining the exact stress diagrams. In standard laboratory experiments with bars initial homogeneous stressed state of the bar near ultimate stress becomes complex nonhomogeneous one with spontaneously appearing flowing neck. Unlike these, inside falling bar we have the tidal gravitational stretching that reaches its maximum at the center of the bar at any moment of time. Due to the volume rather than surface character of tidal forces, the stress diagram obtained by experiments with falling bars, would be independent of the end's stressing and Saint-Venant principle.

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